2C9
Design for seismic and climate changes

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List of lectures

1. Elements of seismology and seismicity I
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3. Dynamic analysis of single-degree-of-freedom systems I
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Finite element method in structural dynamics I
Free vibration analysis

1. FINITE ELEMENT METHOD IN STRUCTURAL DYNAMICS

Statics

$$\mathbf{K}\mathbf{r} = \mathbf{f}$$

Forces do not vary with time

Structural stiffness **K** considered only

$$\mathbf{K} = A_{e=1}^{N_e} \mathbf{K}_e$$

$$\mathbf{K}_e = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV$$

Operator $A_{e=1}^{N_e}$ denotes the direct assembly procedure for assembling the stiffness matrix for each element, e = 1 to N_e, where N_e is the number of elements

Dynamics

$$\mathbf{M}\,\ddot{\mathbf{r}}(t) + \mathbf{K}\,\mathbf{r}(t) = \mathbf{0}$$

Free vibration

No external forces present

Structural stiffness **K** and mass **M** considered

$$\mathbf{M}\,\ddot{\mathbf{r}}(t) + \mathbf{C}\,\dot{\mathbf{r}}(t) + \mathbf{K}\,\mathbf{r}(t) = \mathbf{f}(t)$$

Forced vibration

External forces (usually time dependent) present

Structural stiffness **K,** mass **M** and possibly damping **C** considered

Finite Element Method in structural dynamics

Inertial force (per unit volume)

ρ – mass density

$$\hat{\mathbf{f}}_i = -\rho\ddot{\mathbf{u}} \quad \Rightarrow \quad \mathbf{f}_i = \int_V -\rho\ddot{\mathbf{u}}\,dV$$

Damping force (per unit volume)

ξ – damping coefficient

$$\hat{\mathbf{f}}_d = -\xi\dot{\mathbf{u}} \quad \Rightarrow \quad \mathbf{f}_d = \int_V -\xi\dot{\mathbf{u}}\,dV$$

Principle of virtual displacements

$$\boxed{\delta W_I = \delta W_E}$$

internal virtual work = external virtual work

$$\delta W_I = \int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV$$

$\boldsymbol{\sigma}$ – stress vector $\boldsymbol{\varepsilon}$ – strain vector

$$\delta W_E = \int_V \delta\mathbf{u}^T \hat{\mathbf{f}}_i\, dV + \int_V \delta\mathbf{u}^T \hat{\mathbf{f}}_d\, dV$$

Finite Element Method in structural dynamics

Finite Element context

$$\mathbf{u} = \mathbf{N}\mathbf{r} \qquad \dot{\mathbf{u}} = \mathbf{N}\dot{\mathbf{r}} \qquad \ddot{\mathbf{u}} = \mathbf{N}\ddot{\mathbf{r}}$$

$$\mathbf{u}^T = \mathbf{r}^T\mathbf{N}^T \qquad \delta\mathbf{u}^T = \delta\mathbf{r}^T\mathbf{N}^T$$

$\mathbf{r}$ – nodal displacements
$\mathbf{N}$ – interpolation (shape) functions

Strain – displacement relations

$$\boldsymbol{\varepsilon} = \partial^T \mathbf{u} = \partial^T \mathbf{N}\,\mathbf{r} = \mathbf{B}\,\mathbf{r}$$

$$\delta\boldsymbol{\varepsilon}^T = \delta\mathbf{r}^T\mathbf{B}^T$$

Constitutive relations
(linear elastic material)

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\varepsilon} = \mathbf{D}\,\mathbf{B}\,\mathbf{r}$$

$\mathbf{D}$ – constitutive matrix

Finite Element Method in structural dynamics

Internal virtual work $\delta W_I = \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV = \delta \mathbf{r}^T \int_V \mathbf{B}^T \mathbf{D} \mathbf{B}\, dV\, \mathbf{r}$

External virtual work $\delta W_E = \int_V \delta \mathbf{u}^T \hat{\mathbf{f}}_i\, dV + \int_V \delta \mathbf{u}^T \hat{\mathbf{f}}_d\, dV$

$$= -\delta \mathbf{u}^T \int_V \rho \ddot{\mathbf{u}}\, dV - \delta \mathbf{u}^T \int_V \xi \dot{\mathbf{u}}\, dV$$

$$= -\delta \mathbf{r}^T \int_V \mathbf{N}^T \rho \mathbf{N}\, dV\, \ddot{\mathbf{r}} - \delta \mathbf{r}^T \int_V \mathbf{N}^T \xi \mathbf{N}\, dV\, \dot{\mathbf{r}}$$

$$-\delta W_E + \delta W_I = 0$$

$$\delta \mathbf{r}^T \left(\int_V \mathbf{N}^T \rho \mathbf{N}\, dV\, \ddot{\mathbf{r}} + \int_V \mathbf{N}^T \xi \mathbf{N}\, dV\, \dot{\mathbf{r}} + \int_V \mathbf{B}^T \mathbf{D} \mathbf{B}\, dV\, \mathbf{r} \right) = 0$$

Dynamic equations of equilibrium $\boxed{\mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{0}}$

$\boxed{\mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{f}}$ **f** – nodal forces (external)

Mass matrix
(consistent formulation)

$$\mathbf{M} = \int_V \mathbf{N}^T \rho \mathbf{N} \, dV$$

Damping matrix

$$\mathbf{C} = \int_V \mathbf{N}^T \xi \mathbf{N} \, dV$$

Stiffness matrix

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV$$

Mass matrix - other possible formulations

Diagonal mass matrix — *equal mass lumping* – lumped-mass idealization with equal diagonal terms

special mass lumping - diagonal mass matrix, terms proportional to the consistent mass matrix

Beam element length L, modulus of elasticity E, cross-section area A
moment of inertia (2nd moment of area) I_z, uniform mass m

Starting point – approximation of displacements $\mathbf{u} = \{u, v\}^T$

Exact solution (based on the assumptions adopted)

- Approximation of axial displacements

$$u(x) = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2 = h_1 u_1 + h_2 u_2$$

u_1 – left-end axial displacement
u_2 – right-end axial displacement

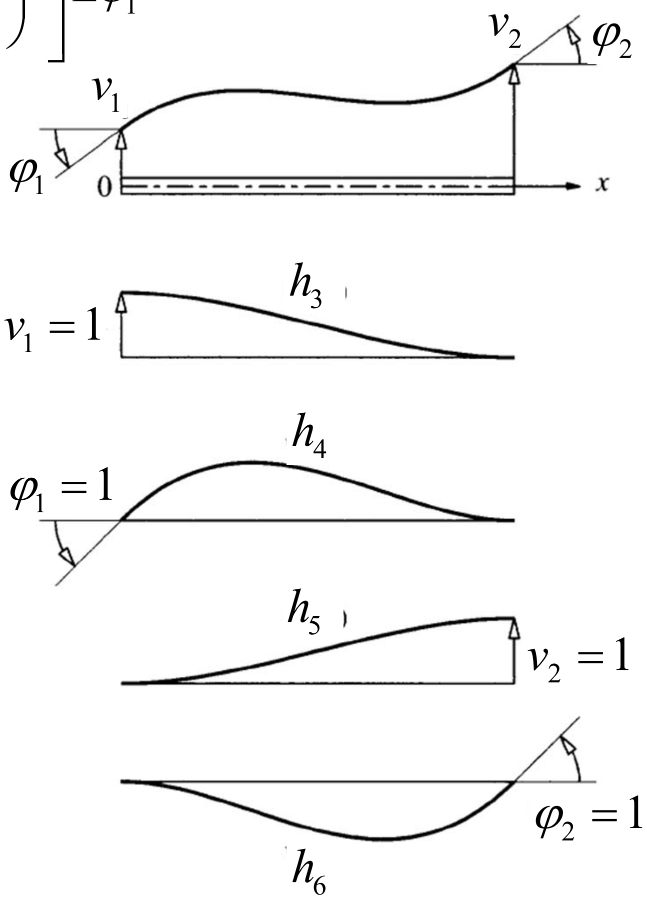
Beam element

- Approximation of deflections

$$v(x) = \left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right] v_1 + \left[\left(\frac{x}{L}\right) - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right] L\varphi_1$$

$$+ \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] v_2 + \left[-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right] L\varphi_2$$

$$v(x) = h_3 v_1 + h_4 \varphi_1 + h_5 v_2 + h_6 \varphi_2$$



v_1 – left-end deflection
v_2 – right-end deflection
φ_1 – left-end rotation
φ_2 – right-end rotation

Beam element

Vector of nodal displacements

$$\mathbf{r} = \{u_1, v_1, \varphi_1, u_2, v_2, \varphi_2\}^T$$

Matrix of interpolation functions

$$\mathbf{N} = \begin{bmatrix} h_1 & 0 & 0 & h_2 & 0 & 0 \\ 0 & h_3 & h_4 & 0 & h_5 & h_6 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{N}\mathbf{r}$$

Constitutive relations

$$\begin{Bmatrix} N_x \\ M_z \end{Bmatrix} = \begin{bmatrix} EA & 0 \\ 0 & EI_z \end{bmatrix} \begin{Bmatrix} \dfrac{du}{dx} \\ \dfrac{d^2v}{dx^2} \end{Bmatrix}$$

→ $\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\varepsilon}$

Strain – displacement relations → $\boldsymbol{\varepsilon} = \partial^T \mathbf{u} = \partial^T \mathbf{N}\,\mathbf{r} = \mathbf{B}\,\mathbf{r}$

$$\mathbf{B} = \begin{bmatrix} \dfrac{dh_1}{dx} & 0 & 0 & \dfrac{dh_2}{dx} & 0 & 0 \\ 0 & \dfrac{d^2h_3}{dx^2} & \dfrac{d^2h_4}{dx^2} & 0 & \dfrac{d^2h_5}{dx^2} & \dfrac{d^2h_6}{dx^2} \end{bmatrix}$$

Beam element

Element stiffness matrix

$$\mathbf{K} = \int_0^L \mathbf{B}^T \mathbf{D} \mathbf{B} \, dx$$

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}$$

Element mass matrix
consistent formulation

$$\mathbf{M} = \int_0^L \mathbf{N}^T m \, \mathbf{N} \, dx$$

$$\mathbf{M} = \frac{mL}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ & 156 & 22L & 0 & 54 & -13L \\ & & 4L^2 & 0 & 13L & -3L^2 \\ & & & 140 & 0 & 0 \\ & & & & 156 & -22L \\ sym & & & & & 4L^2 \end{bmatrix}$$

Element mass matrix
other possible approximation
of deflections

$$v(x) = \left(1 - \frac{x}{L}\right) v_1 + \frac{x}{L} v_2 = h_1 v_1 + h_2 v_2$$

$$\mathbf{M} = \frac{mL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ & 140 & 0 & 0 & 70 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 140 & 0 & 0 \\ & & & & 140 & 0 \\ sym & & & & & 0 \end{bmatrix}$$

Element mass matrix
diagonal mass matrix

$$v(x) = h_1 v_1 + h_2 v_2$$

where $h_1 = 1$ for $x \leq l/2$
$h_1 = 0$ for $x > l/2$

$h_2 = 0$ for $x < l/2$
$h_2 = 1$ for $x \geq l/2$

$$\mathbf{M} = \frac{mL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ sym & & & & & 0 \end{bmatrix}$$

Comparison of Finite Element and exact solution

TABLE 17.10.1 NATURAL FREQUENCIES OF A UNIFORM CANTILEVER BEAM: CONSISTENT-MASS FINITE ELEMENT AND EXACT SOLUTIONS

	Number of Finite Elements, N_e					
Mode	1	2	3	4	5	Exact
1	3.53273	3.51772	3.51637	3.51613	3.51606	3.51602
2	34.8069	22.2215	22.1069	22.0602	22.0455	22.0345
3		75.1571	62.4659	62.1749	61.9188	61.6972
4		218.138	140.671	122.657	122.320	120.902
5			264.743	228.137	203.020	199.860
6			527.796	366.390	337.273	298.556
7				580.849	493.264	416.991
8				953.051	715.341	555.165
9					1016.20	713.079
10					1494.88	890.732

Source: R. R. Craig, Jr., *Structural Dynamics*, Wiley, New York, 1981.

SUSTAINABLE STEEL AND TIMBER CONSTRUCTIONS

Comparison of Finite Element and exact solution

TABLE 17.10.2 NATURAL FREQUENCIES OF A UNIFORM CANTILEVER BEAM: LUMPED-MASS FINITE ELEMENT AND EXACT SOLUTIONS

	Number of Finite Elements, N_e					
Mode	1	2	3	4	5	Exact
1	2.44949	3.15623	3.34568	3.41804	3.45266	3.51602
2		16.2580	18.8859	20.0904	20.7335	22.0345
3			47.0284	53.2017	55.9529	61.6972
4				92.7302	104.436	120.902
5					153.017	199.860

Source: R. R. Craig, Jr., *Structural Dynamics*, Wiley, New York, 1981.

Accuracy of FE analysis is improved by increasing number of DOFs

2. FREE VIBRATION

natural vibration frequencies and modes

$$\mathbf{M}\,\ddot{\mathbf{r}}(t) + \mathbf{K}\,\mathbf{r}(t) = \mathbf{0}$$

equation of motion
for undamped free vibration of MDOF system

$$\mathbf{r}(t) = \boldsymbol{\phi}_n (A_n \cos \omega_n t + B_n \sin \omega_n t)$$

motions of a system in free vibrations are simple harmonic

$$\ddot{\mathbf{r}}(t) = -\omega_n^2 \mathbf{r}(t)$$

$$\left(\mathbf{K} - \omega_n^2 \mathbf{M}\right)\boldsymbol{\phi}_n = \mathbf{0}$$

free vibration equation, eigenvalue equation

- $\boldsymbol{\phi}_n$ - natural mode
- ω_n - natural frequency

$$\det\left(\mathbf{K} - \omega_n^2 \mathbf{M}\right) = 0$$

nontrivial solution condition
frequency equation (polynomial of order *N)*
not a practical method for larger systems

3. RAYLEIGH – RITZ METHOD

general technique for reducing the number of degrees of freedom

$$\left(\mathbf{K} - \omega_n^2 \mathbf{M}\right)\boldsymbol{\phi}_n = \mathbf{0}$$

$$\left(\mathbf{K} - \lambda \mathbf{M}\right)\boldsymbol{\phi} = \mathbf{0}$$

$$\boldsymbol{\phi}^T \mathbf{K} \boldsymbol{\phi} = \lambda \boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi}$$

$$\boxed{\lambda = \frac{\boldsymbol{\phi}^T \mathbf{K} \boldsymbol{\phi}}{\boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi}}}$$

Rayleigh's quotient

Properties

1. When $\boldsymbol{\phi}$ is an eigenvector $\boldsymbol{\phi}_n$, Rayleigh's quotient is equal to the corresponding eigenvalue $\lambda = \omega_n^2$
2. Rayleigh's quotient is bounded between the smallest and largest eigenvalues

$$\omega_1^2 \le \lambda_1 \quad \omega_2^2 \le \lambda_2 \quad .... \quad \omega_n^2 \le \lambda_n$$

Rayleigh – Ritz method

$$\mathbf{u}(t) = \boldsymbol{\Psi}\,\mathbf{z}(t)$$

$$\boldsymbol{\phi} = \boldsymbol{\Psi}\,\mathbf{z}$$

displacements or modal shapes are expressed as a linear combination of shape vectors ψ_j , j = 1,2…J <N

Ritz vectors make up the columns of N x J matrix $\boldsymbol{\Psi}$

$\mathbf{z}$ is a vector of J generalized coordinates

substituting the Ritz transformation in a system of *N* equations of motion

$$\mathbf{K}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{s}p(t)$$

we obtain a system of *J* equations in generalized coordinates

$$\boxed{\tilde{\mathbf{m}}\ddot{\mathbf{z}} + \tilde{\mathbf{c}}\dot{\mathbf{z}} + \tilde{\mathbf{k}}\mathbf{z} = \tilde{\mathbf{s}}p(t)}$$

$$\tilde{\mathbf{m}} = \boldsymbol{\Psi}^T\mathbf{M}\,\boldsymbol{\Psi} \qquad \tilde{\mathbf{c}} = \boldsymbol{\Psi}^T\mathbf{C}\,\boldsymbol{\Psi} \qquad \tilde{\mathbf{k}} = \boldsymbol{\Psi}^T\mathbf{K}\,\boldsymbol{\Psi} \qquad \tilde{\mathbf{s}} = \boldsymbol{\Psi}^T\mathbf{s}$$

Ritz transformation has made it possible to reduce original set of *N* equations in the nodal displacement $\mathbf{u}$ to a smaller set of *J* equations in the generalized coordinates $\mathbf{z}$

Rayleigh – Ritz method

substituting the Ritz transformation in a Rayleigh's quotient

$$\lambda = \frac{\boldsymbol{\phi}^T \mathbf{K} \boldsymbol{\phi}}{\boldsymbol{\phi}^T \mathbf{M} \boldsymbol{\phi}} = \frac{\mathbf{z}^T \boldsymbol{\Psi}^T \mathbf{K} \boldsymbol{\Psi} \mathbf{z}}{\mathbf{z}^T \boldsymbol{\Psi}^T \mathbf{M} \boldsymbol{\Psi} \mathbf{z}} = \rho(z)$$

and using Rayleigh's stationary condition

$$\frac{\partial \rho(z)}{\partial z_i} = 0 \qquad i = 1, 2, \ldots J$$

we obtain the reduced eigenvalue problem

$$\boxed{\tilde{\mathbf{k}}\mathbf{z} = \rho \tilde{\mathbf{m}}\mathbf{z}} \qquad \Rightarrow \quad \rho, \mathbf{z}$$

solution of all eigenvalues and eigenmodes
e.g. Jacobi's method of rotations

$$\tilde{\mathbf{m}} = \boldsymbol{\Psi}^T \mathbf{M} \boldsymbol{\Psi} \qquad \tilde{\mathbf{k}} = \boldsymbol{\Psi}^T \mathbf{K} \boldsymbol{\Psi}$$

$$\omega_1^2 \le \rho_1 \quad \omega_2^2 \le \rho_2 \quad \ldots. \quad \omega_J^2 \le \rho_J \qquad\qquad \boldsymbol{\phi}_i = \boldsymbol{\psi}_i \mathbf{z}_i$$

Selection of Ritz vectors

GENERATION OF FORCE-DEPENDENT RITZ VECTORS

1. Determine the first vector, ψ_1.

a. Determine $\mathbf{y}_1$ by solving: $\mathbf{k}\mathbf{y}_1 = \mathbf{s}$.

b. Normalize $\mathbf{y}_1$: $\psi_1 = \mathbf{y}_1 \div (\mathbf{y}_1^T \mathbf{m} \mathbf{y}_1)^{1/2}$.

2. Determine additional vectors, $\psi_n, n = 2, 3, \ldots, J$.

a. Determine $\mathbf{y}_n$ by solving: $\mathbf{k}\mathbf{y}_n = \mathbf{m}\psi_{n-1}$.

b. Orthogonalize $\mathbf{y}_n$ with respect to previous $\psi_1, \psi_2, \ldots, \psi_{n-1}$ by repeating the following steps for $i = 1, 2, \ldots, n-1$:

- $a_{in} = \psi_i^T \mathbf{m} \mathbf{y}_n$.
- $\hat{\psi}_n = \mathbf{y}_n - a_{in}\psi_i$.
- $\mathbf{y}_n = \hat{\psi}_n$.

c. Normalize $\hat{\psi}_n$: $\psi_n = \hat{\psi}_n \div (\hat{\psi}_n^T \mathbf{m} \hat{\psi}_n)^{1/2}$.

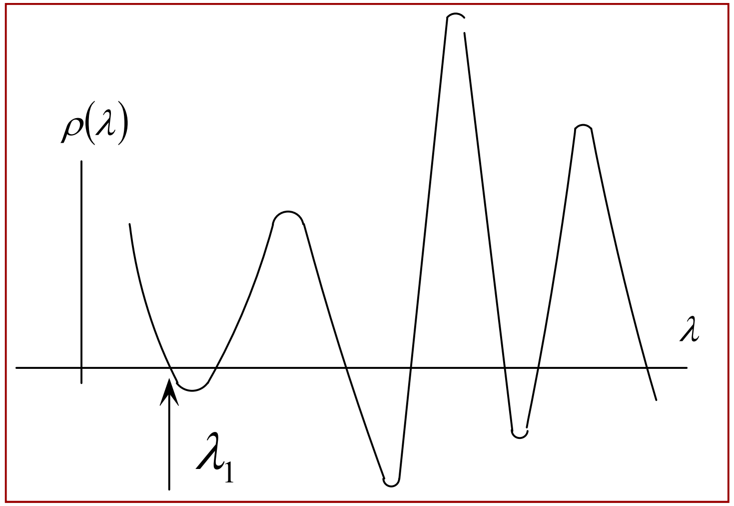
Inverse iteration (Stodola-Vianello)

Algorithm to determine
the **lowest** eigenvalue

$$\mathbf{K}\overline{\mathbf{x}}_{k+1} = \mathbf{M}\mathbf{x}_k \Rightarrow \overline{\mathbf{x}}_{k+1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{x}_k$$

$$\mathbf{x}_{k+1} = \frac{\overline{\mathbf{x}}_{k+1}}{(\overline{\mathbf{x}}_{k+1}^T \mathbf{M}\overline{\mathbf{x}}_{k+1})^{1/2}}$$

$$\overline{\mathbf{x}}_{k+1} \Rightarrow \boldsymbol{\phi}_1 \quad as \quad k \rightarrow \infty$$



Vector iteration techniques

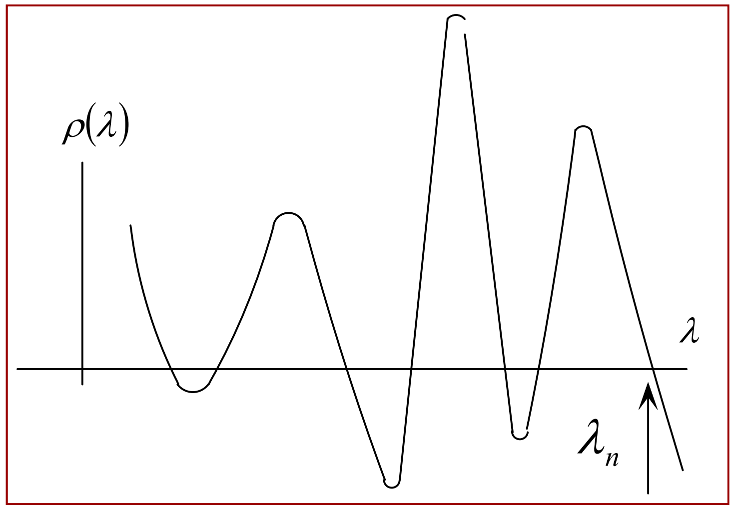
Forward iteration (Stodola-Vianello)

Algorithm to determine
the **highest** eigenvalue

$$\mathbf{M}\overline{\mathbf{x}}_{k+1} = \mathbf{K}\mathbf{x}_k \Rightarrow \overline{\mathbf{x}}_{k+1} = \mathbf{M}^{-1}\mathbf{K}\mathbf{x}_k$$

$$\mathbf{x}_{k+1} = \frac{\overline{\mathbf{x}}_{k+1}}{(\overline{\mathbf{x}}_{k+1}^T \mathbf{M}\overline{\mathbf{x}}_{k+1})^{1/2}}$$

$$\overline{\mathbf{x}}_{k+1} \Rightarrow \boldsymbol{\phi}_n \quad as \quad k \rightarrow \infty$$



5. INVERSE VECTOR ITERATION METHOD

1. Starting vector $\mathbf{x}_1$ – arbitrary choice

$$\mathbf{R}_1 = \mathbf{M}\mathbf{x}_1$$

$$\mathbf{K}\overline{\mathbf{x}}_2 = \mathbf{R}_1$$

2. $\boxed{\mathbf{K}\overline{\mathbf{x}}_{k+1} = \mathbf{M}\mathbf{x}_k}$ $\Rightarrow \overline{\mathbf{x}}_{k+1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{x}_k$

3. $\lambda^{(k+1)} = \dfrac{\overline{\mathbf{x}}_{k+1}^T \mathbf{K}\overline{\mathbf{x}}_{k+1}}{\overline{\mathbf{x}}_{k+1}^T \mathbf{M}\overline{\mathbf{x}}_{k+1}}$ (Rayleigh's quotient)

4. $\dfrac{\left|\lambda^{(k+1)} - \lambda^{(k)}\right|}{\lambda^{(k+1)}} \leq tol$

5. If convergence criterion is not satisfied, normalize

$$\mathbf{x}_{(k+1)} = \frac{\overline{\mathbf{x}}_{k+1}}{\left(\overline{\mathbf{x}}_{k+1}^T \mathbf{M}\overline{\mathbf{x}}_{k+1}\right)^{1/2}}$$

and go back to 2. and set $k = k+1$

Inverse vector iteration method

6. For the last iteration (k+1), when convergence is satisfied

$$\lambda_1 = \omega_1^2 = \lambda^{(k+1)} \qquad \boldsymbol{\phi}_1 = \mathbf{x}_{(k+1)} = \frac{\overline{\mathbf{x}}_{k+1}}{\left(\overline{\mathbf{x}}_{k+1}^T \mathbf{M} \overline{\mathbf{x}}_{k+1}\right)^{1/2}}$$

Gramm-Schmidt orthogonalisation
to progress to other than limiting eigenvalues (highest and lowest)

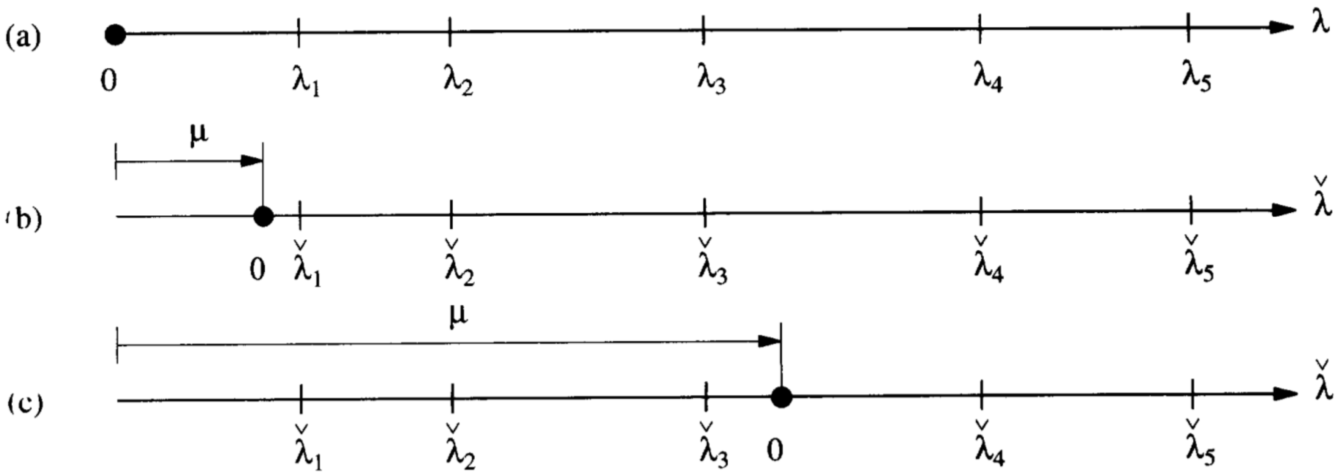
evaluation of $\boldsymbol{\phi}_{n+1}$
correction of trial vector $\mathbf{x}$

$$\tilde{\mathbf{x}} = \mathbf{x} - \sum_{j=1}^{n} \frac{\boldsymbol{\phi}_j^T \mathbf{M} \mathbf{x}}{\boldsymbol{\phi}_j^T \mathbf{M} \boldsymbol{\phi}_j} \boldsymbol{\phi}_j$$

Inverse vector iteration method

Vector iteration with shift μ

the shifting concept enables computation of any eigenpair



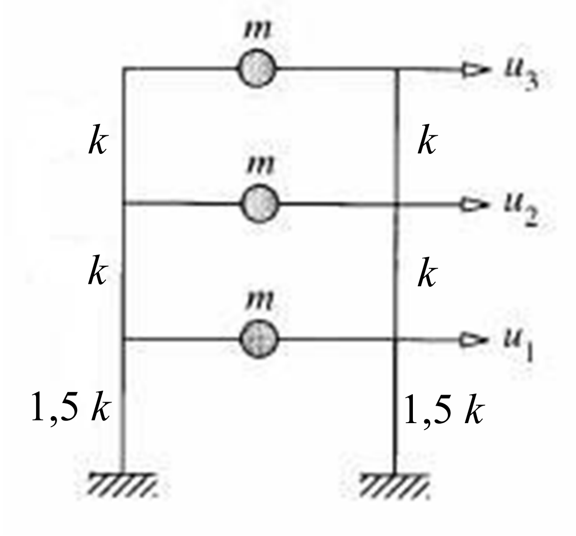
$$\lambda = \breve{\lambda} + \mu \qquad \breve{\mathbf{K}}\boldsymbol{\phi} = \breve{\lambda}\mathbf{M}\boldsymbol{\phi} \qquad \breve{\mathbf{K}} = \mathbf{K} - \mu\mathbf{M}$$

eigenvectors of the two eigenvalue problems are the same

inverse vector iteration converge to the eigenvalue closer to the shifted origin, e.g. to $\breve{\lambda}_3$, see (c)

Inverse vector iteration method

Example – 3DOF frame, 1st natural frequency



flexibility matrix

$$\mathbf{K}^{-1} = \boldsymbol{\delta} = \frac{1}{6k}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 8 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$\boldsymbol{\delta}\mathbf{M} = \frac{m}{6k}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 8 \end{bmatrix}$$

$$\mathbf{K}\overline{\mathbf{x}}_{k+1} = \mathbf{M}\mathbf{x}_k \Rightarrow \quad \overline{\mathbf{x}}_{k+1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{x}_k = \delta\mathbf{M}\mathbf{x}_k$$

$$\mathbf{x}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$ starting vector

Inverse vector iteration method

normalization $u_1 = 1$

1st iteration

$$\bar{\mathbf{x}}_1 = \boldsymbol{\delta}\mathbf{M}\mathbf{x}_0 = \frac{m}{6k}\begin{Bmatrix} 6 \\ 12 \\ 15 \end{Bmatrix} \qquad \rho^{(1)} = \frac{\bar{\mathbf{x}}_1^T \mathbf{M}\mathbf{x}_0}{\bar{\mathbf{x}}_1^T \mathbf{M}\bar{\mathbf{x}}_1} = 0{,}489\frac{k}{m} \qquad \mathbf{x}_1 = \begin{Bmatrix} 1 \\ 2 \\ 2{,}5 \end{Bmatrix}$$

2nd iteration

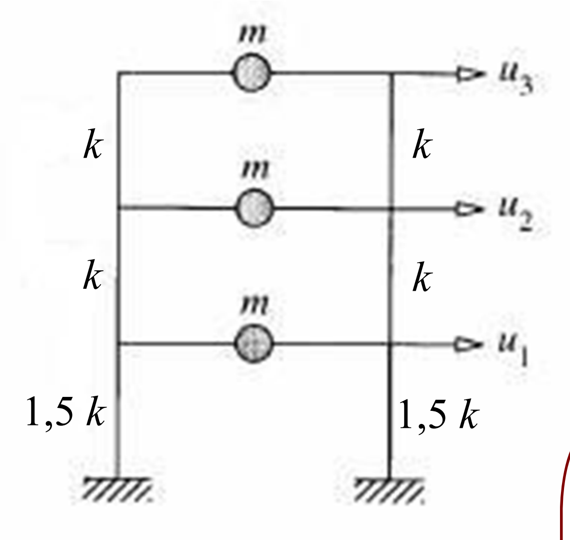
$$\bar{\mathbf{x}}_2 = \boldsymbol{\delta}\mathbf{M}\mathbf{x}_1 = \frac{m}{6k}\begin{Bmatrix} 11 \\ 24{,}5 \\ 32 \end{Bmatrix} \qquad \rho^{(2)} = \frac{\bar{\mathbf{x}}_2^T \mathbf{M}\mathbf{x}_1}{\bar{\mathbf{x}}_2^T \mathbf{M}\bar{\mathbf{x}}_2} = 0{,}481\frac{k}{m} \qquad \mathbf{x}_2 = \begin{Bmatrix} 1 \\ 2{,}23 \\ 2{,}91 \end{Bmatrix}$$

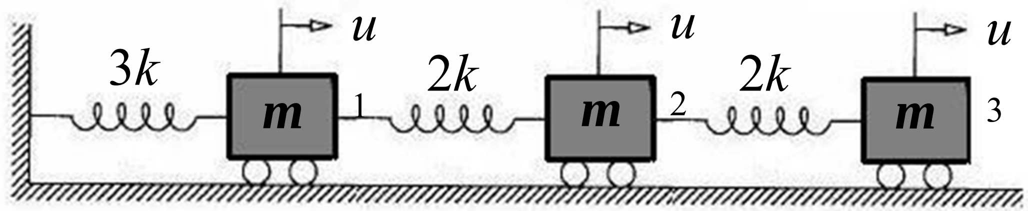
3rd iteration

$$\bar{\mathbf{x}}_3 = \boldsymbol{\delta}\mathbf{M}\mathbf{x}_2 = \frac{m}{6k}\begin{Bmatrix} 12{,}28 \\ 27{,}70 \\ 36{,}43 \end{Bmatrix} \qquad \rho^{(3)} = \frac{\bar{\mathbf{x}}_3^T \mathbf{M}\mathbf{x}_2}{\bar{\mathbf{x}}_3^T \mathbf{M}\bar{\mathbf{x}}_3} = 0{,}481\frac{k}{m} \qquad \mathbf{x}_3 = \begin{Bmatrix} 1 \\ 2{,}26 \\ 2{,}97 \end{Bmatrix}$$

$$\rightarrow \quad \omega_1 \cong \sqrt{\rho^{(3)}} = 0{,}694\sqrt{\frac{k}{m}} \qquad \boldsymbol{\phi}_1 \cong \mathbf{x}_3$$

Appendix – flexibility matrix





$$\rightarrow F_1 = 1$$

$$\delta_{11} = \frac{1}{3k} \quad \delta_{21} = \frac{1}{3k} \quad \delta_{31} = \frac{1}{3k}$$

$$\rightarrow F_2 = 1$$

$$\delta_{12} = \frac{1}{3k} \quad \delta_{22} = \frac{1}{3k} + \frac{1}{2k} \quad \delta_{32} = \frac{1}{3k} + \frac{1}{2k}$$

$$\rightarrow F_3 = 1$$

$$\delta_{13} = \frac{1}{3k} \quad \delta_{23} = \frac{1}{3k} + \frac{1}{2k} \quad \delta_{33} = \frac{1}{3k} + \frac{1}{2k} + \frac{1}{2k}$$

$$\boldsymbol{\delta} = \frac{1}{6k}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 8 \end{bmatrix}$$

6. SUBSPACE ITERATION METHOD

efficient method for eigensolution of large systems when only the lower modes are of interest

similar to inverse iteration method - iteration is performed simultaneously on a number of trial vectors *m*

m – smaller of 2*p* and *p*+8, *p* is the number of modes to be determined (*p* is usually much less then *N,* number of DOF)

1. Staring vectors $\mathbf{X}_1$

$$\mathbf{R}_1 = \mathbf{M}\mathbf{X}_1$$

$$\mathbf{K}\bar{\mathbf{X}}_2 = \mathbf{R}_1$$

2. Subspace iteration

a) $$\mathbf{K}\bar{\mathbf{X}}_{k+1} = \mathbf{M}\mathbf{X}_k$$

b) Ritz transformation $$\tilde{\mathbf{K}}_{k+1} = \bar{\mathbf{X}}_{k+1}^T\mathbf{K}\bar{\mathbf{X}}_{k+1} \qquad \tilde{\mathbf{M}}_{k+1} = \bar{\mathbf{X}}_{k+1}^T\mathbf{M}\bar{\mathbf{X}}_{k+1}$$

Subspace iteration method

2. Subspace iteration *cont.*

c) Reduced eigenvalue problem (*m* eigenvalues)

$$\tilde{\mathbf{K}}_{k+1}\mathbf{Q}_{k+1} = \mathbf{\Omega}^2_{k+1}\tilde{\mathbf{M}}_{k+1}\mathbf{Q}_{k+1}$$

d) New vectors

$$\mathbf{X}_{k+1} = \bar{\mathbf{X}}_{k+1}\mathbf{Q}_{k+1}$$

e) Go back to 2 a)

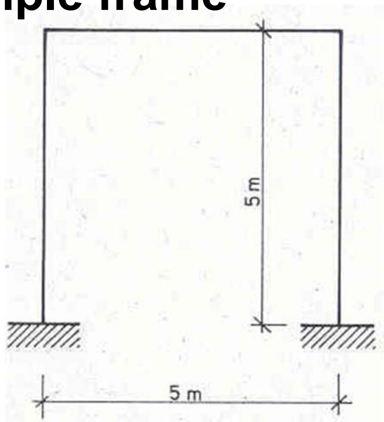
3. Sturm sequence check

verification that the required eigenvalues and vectors have been calculated - i.e. first *p* eigenpairs

Subspace iteration method - combines vector iteration method with the transformation method

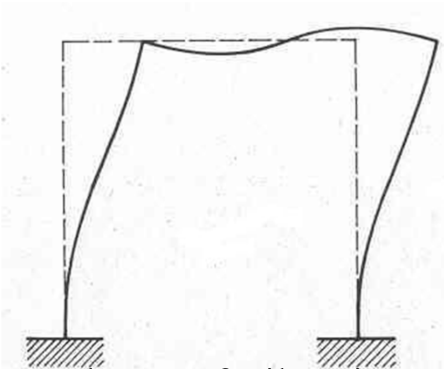
7. EXAMPLES

Simple frame

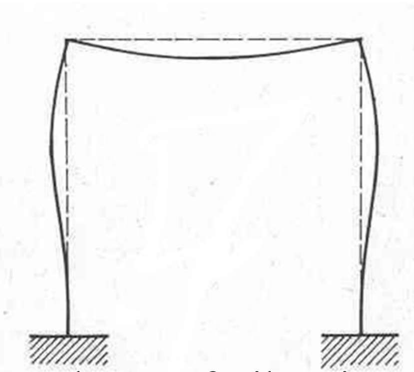


EI = 32 000 kNm^2
μ = 252 kgm^{-1}

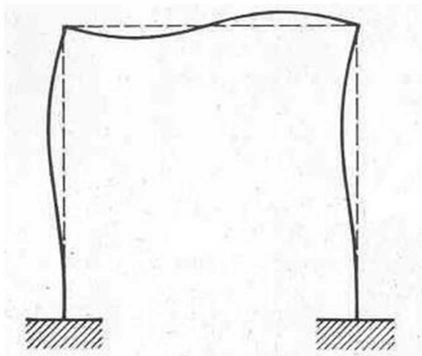
	exact solution	3 elem	6 elem	12 elem
f_1 [Hz]	7,270	7,282	7,270	7,270
f_2 [Hz]	28,693	34,285	28,845	28,711
f_3 [Hz]	46,799	74,084	47,084	46,854



1. shape of vibration



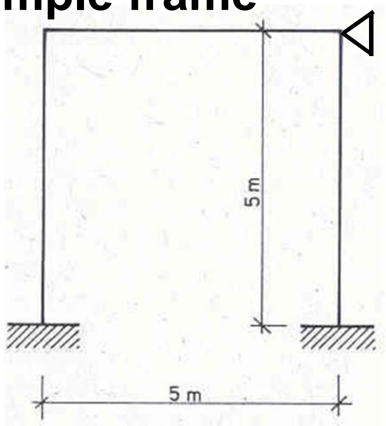
2. shape of vibration



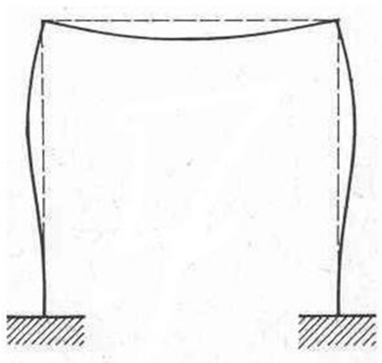
3. shape of vibration

Examples

Simple frame



EI = 32 000 kNm^2
μ = 252 kgm^{-1}

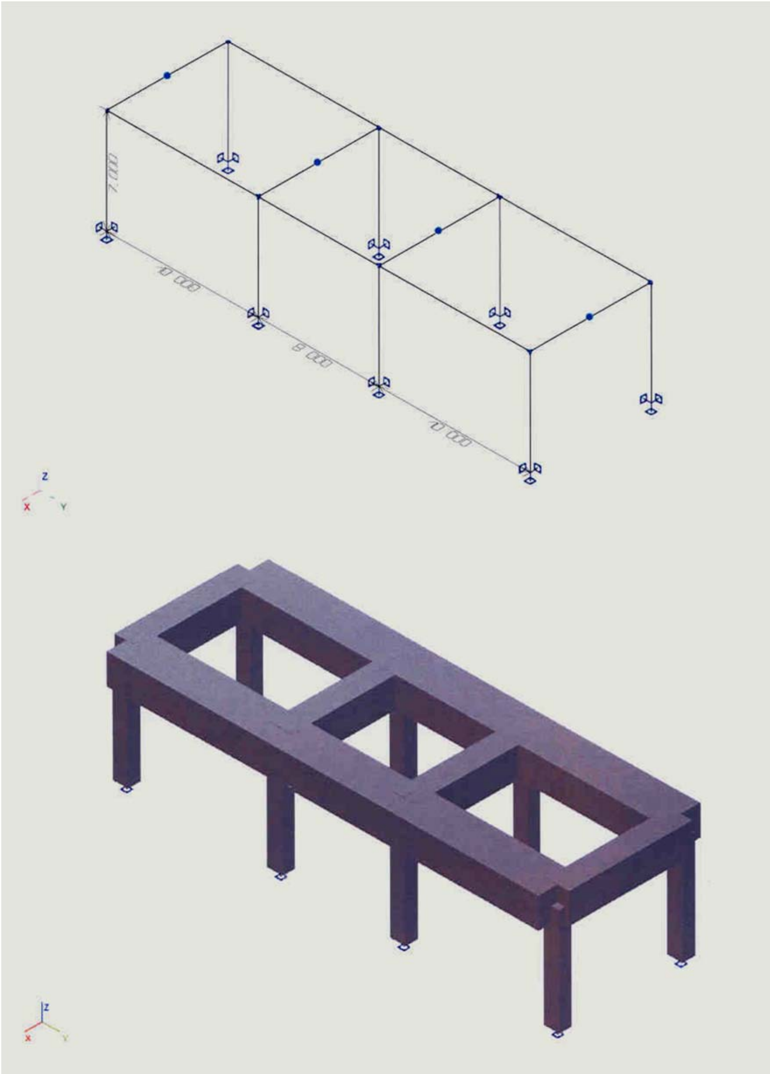


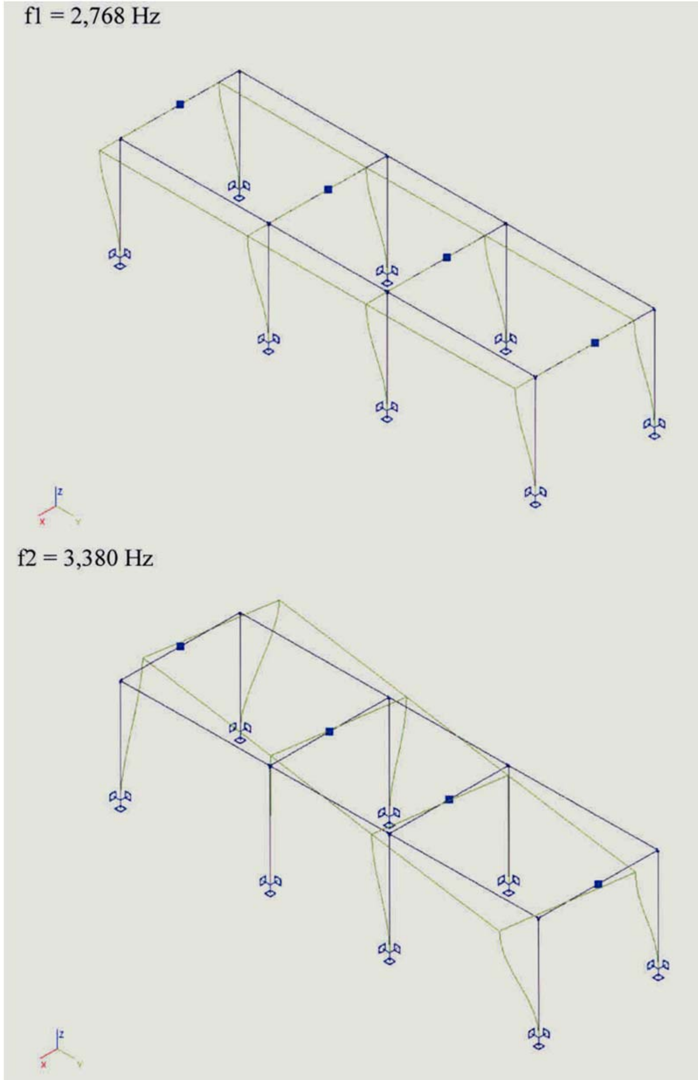
1. shape of vibration

	exact solution	3 elem	6 elem
f_1 [Hz]	28,662	34,285	28,845
f_2 [Hz]	41,863	65,623	42,393
f_3 [Hz]	50,653	-	51,474

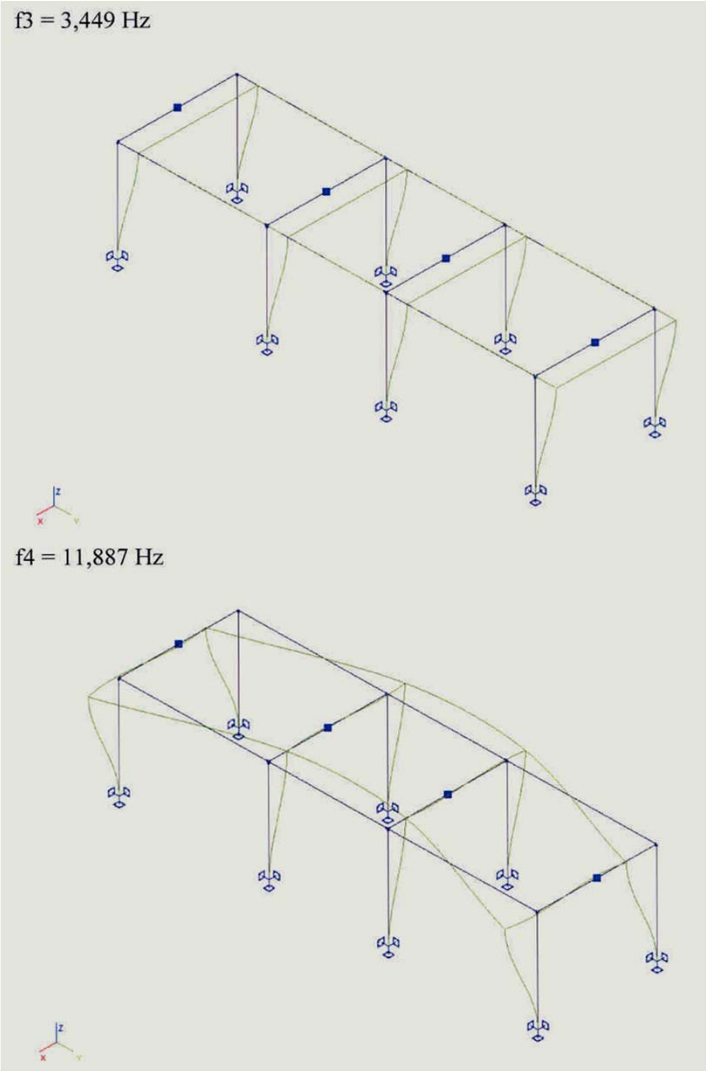
Recommendation:
it is good to divide beams into (at least) 2 elements in the dynamic FE analysis

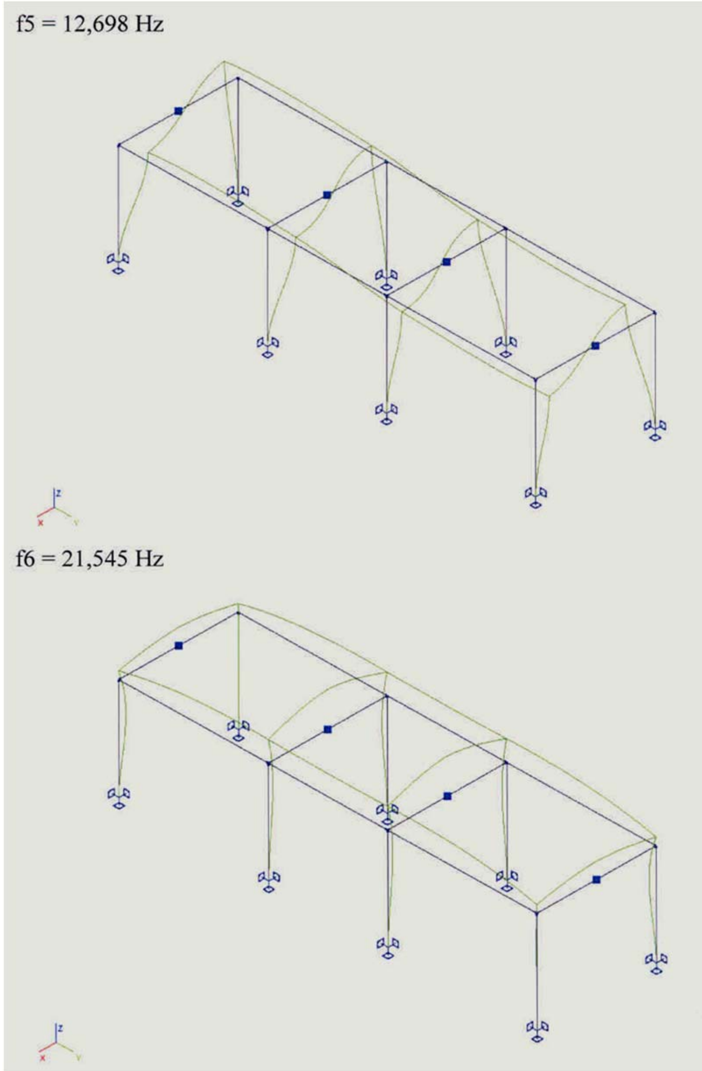
Examples – 3-D frame (turbomachinery frame foundation)



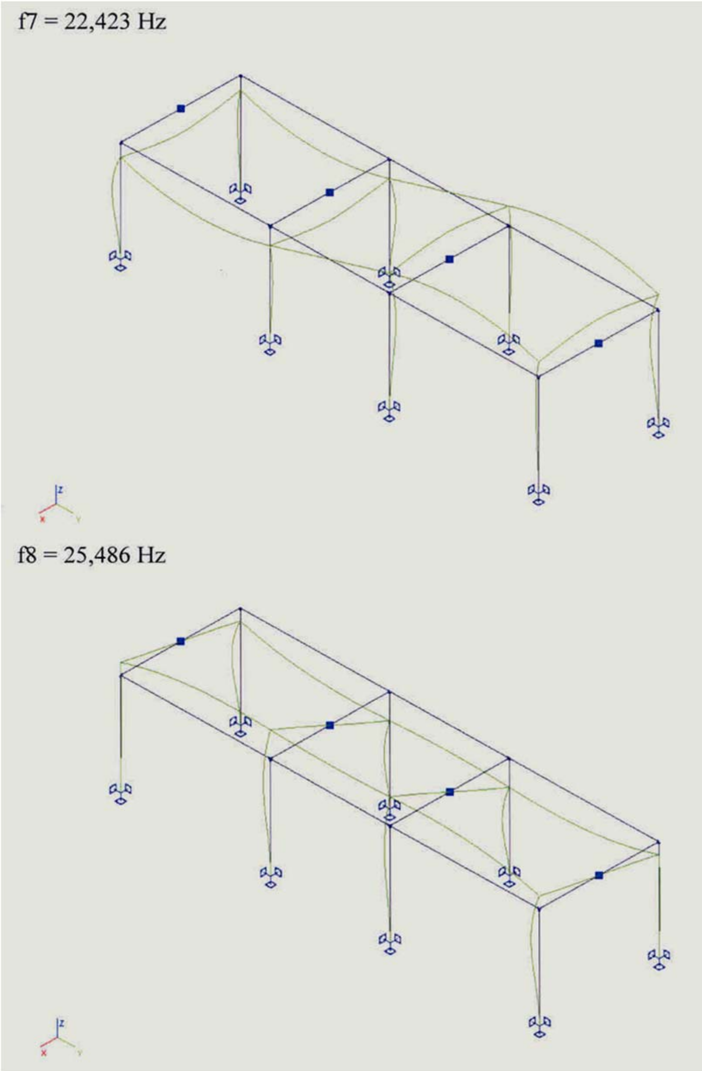


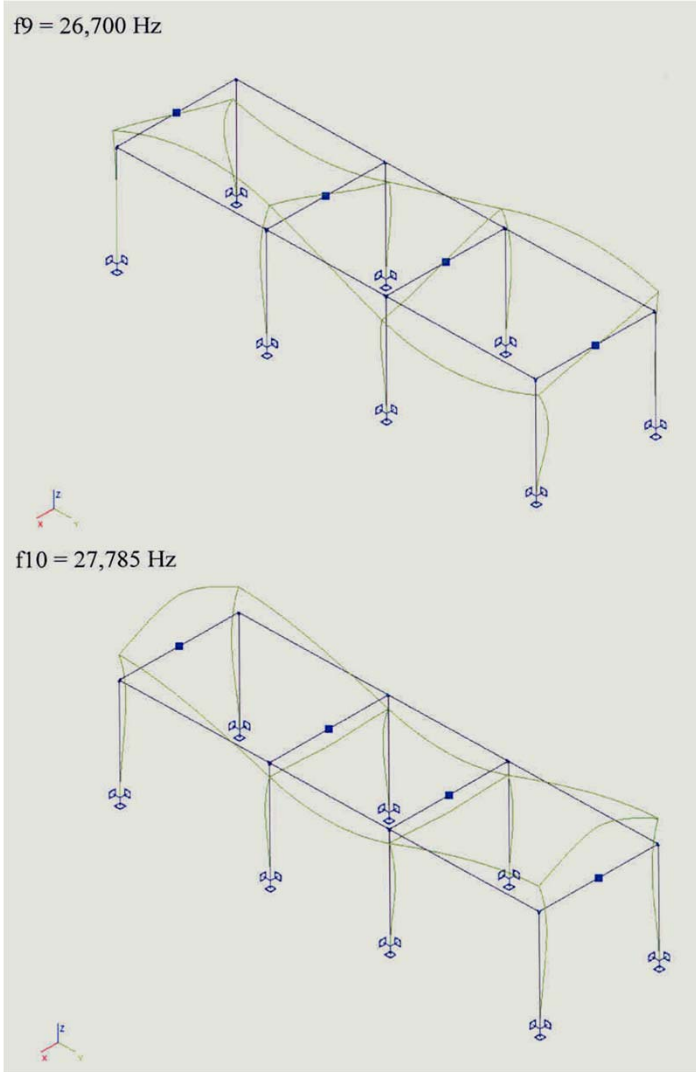
Examples – 3-D frame (turbo-machinery frame foundation)



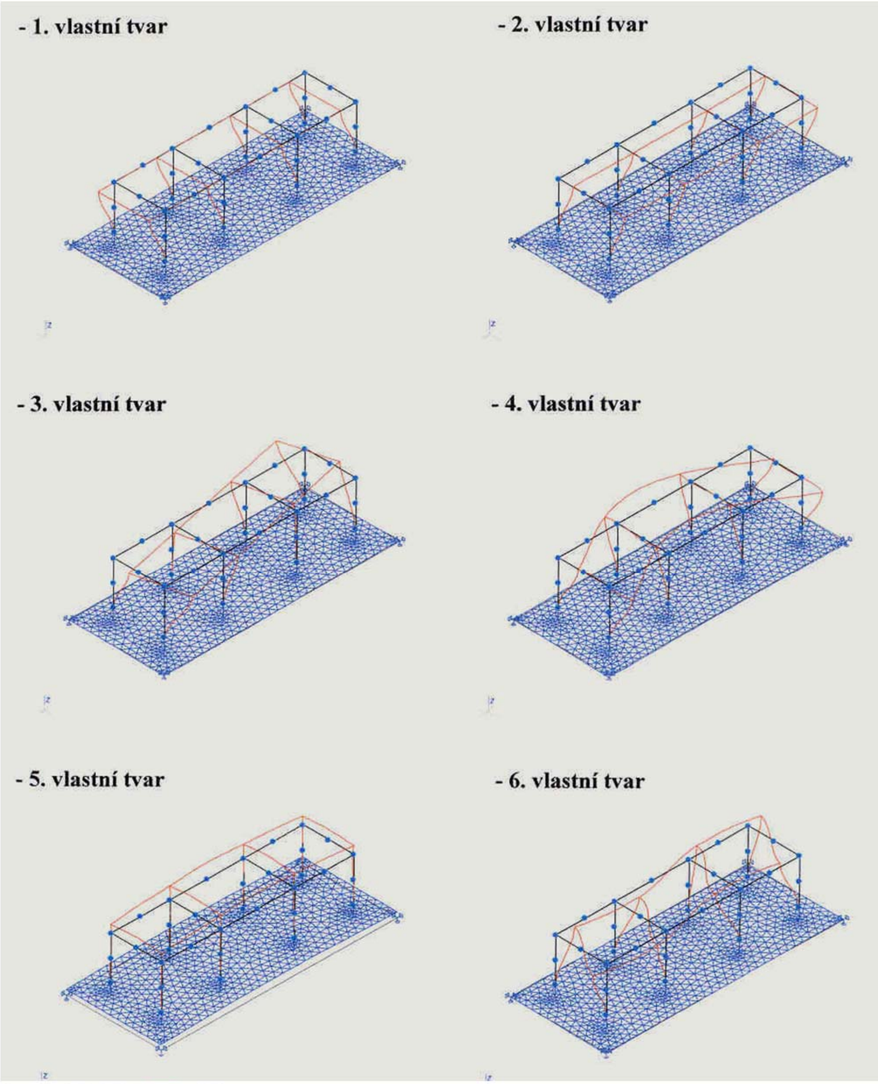


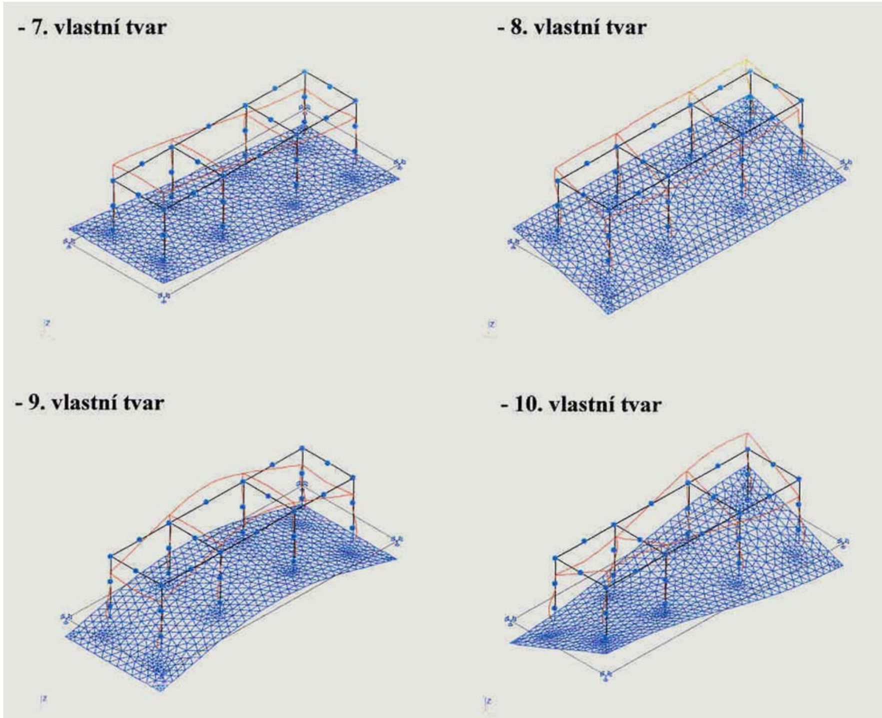
Examples – 3-D frame (turbo-machinery frame foundation)



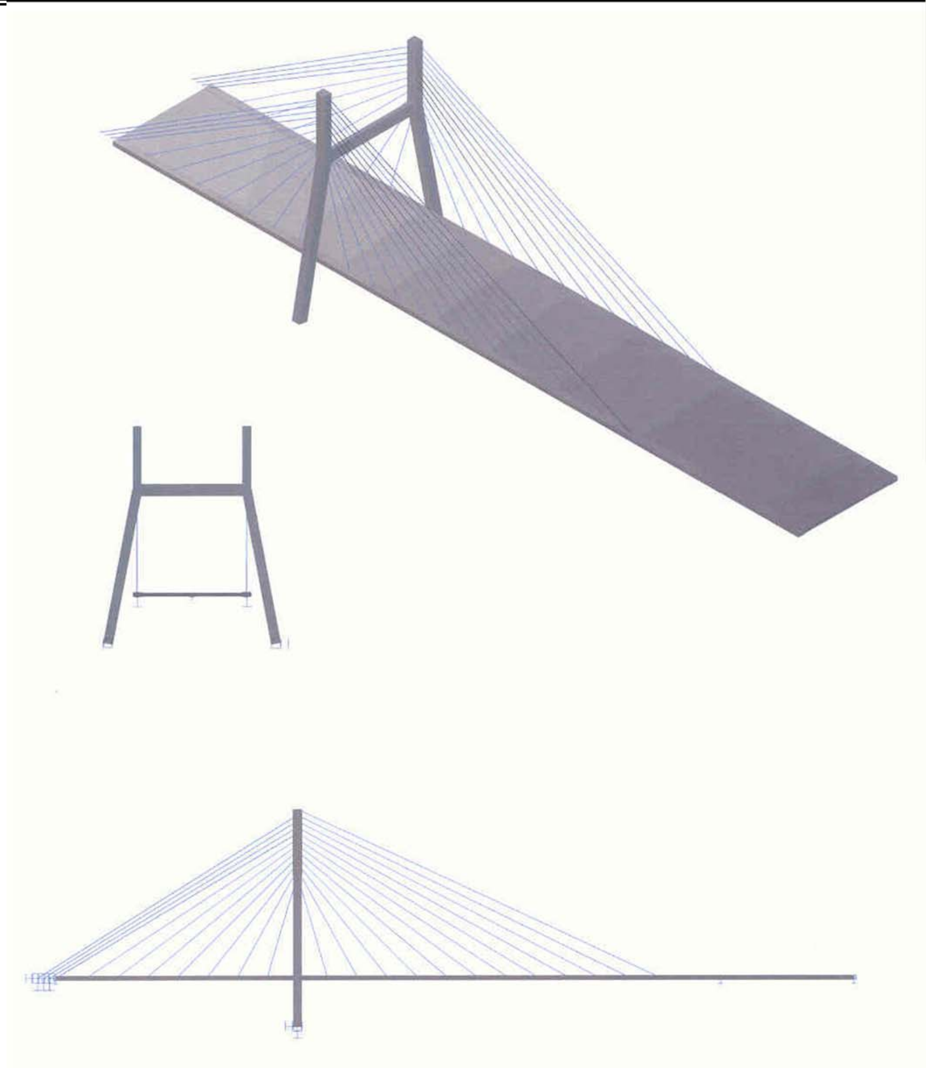


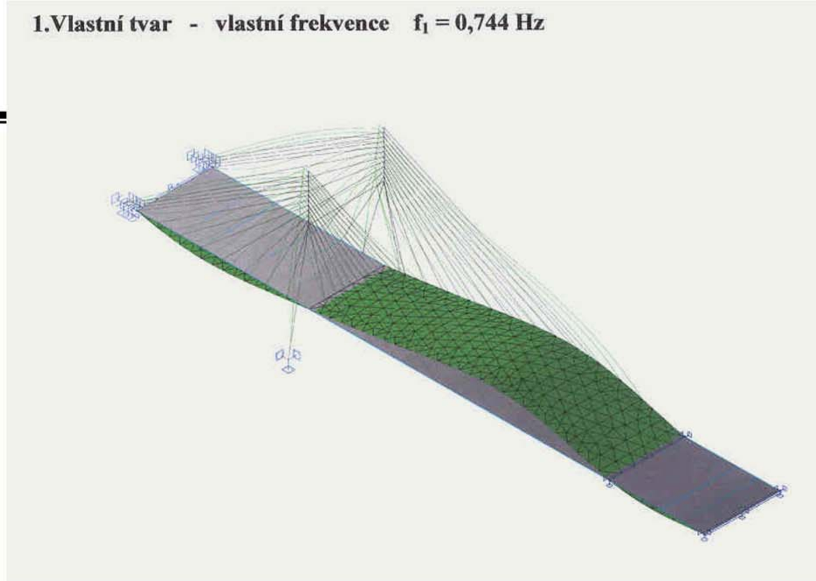
Examples – 3-D frame + plate + elastic foundation

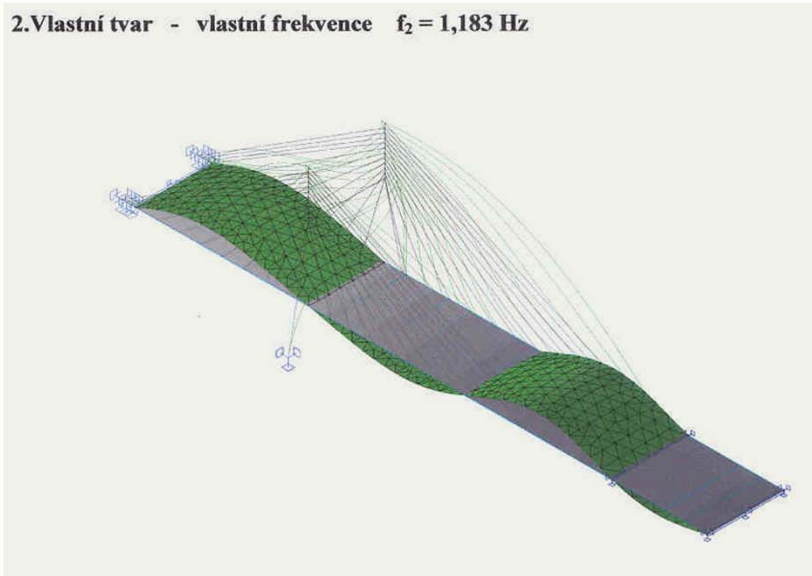




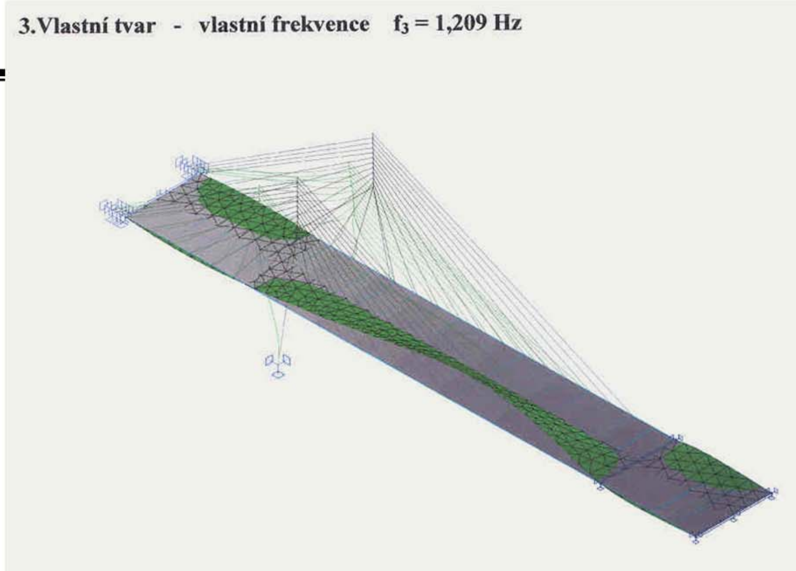
Examples – cable-stayed bridge

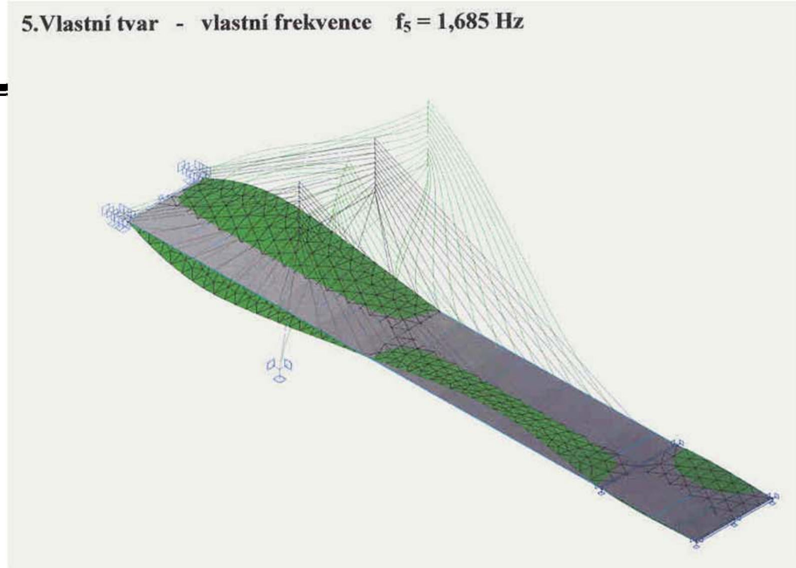


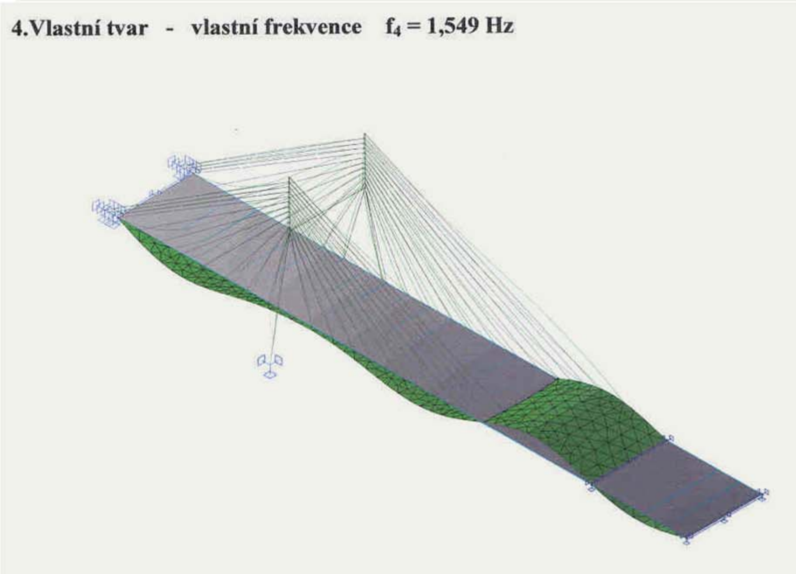


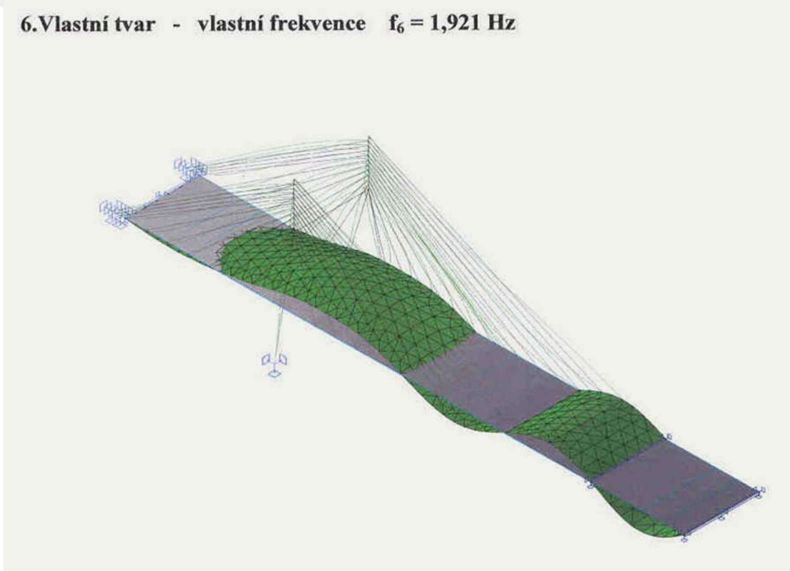


Examples – cable-stayed bridge



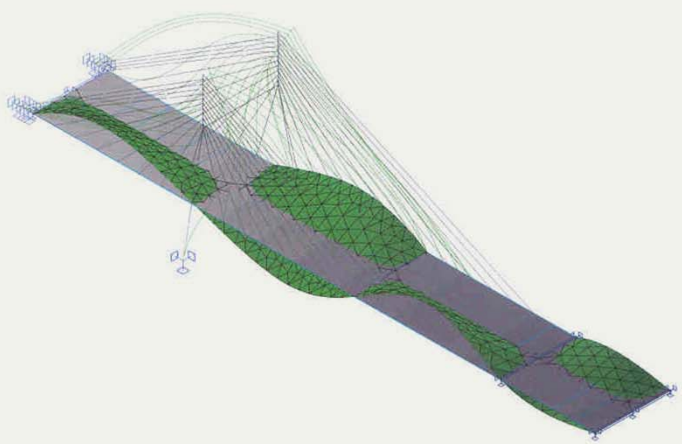




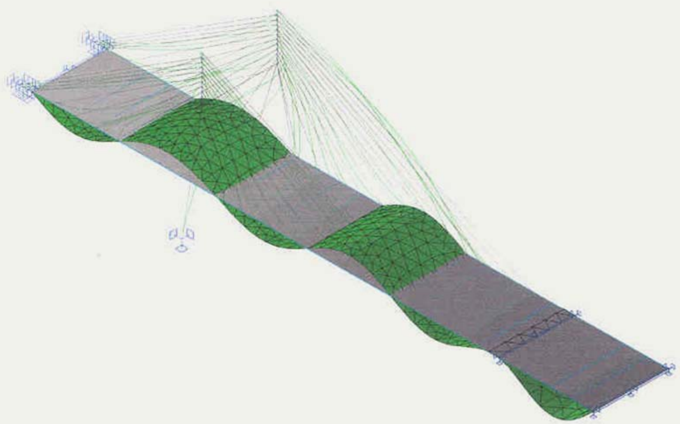


Examples – cable-stayed bridge

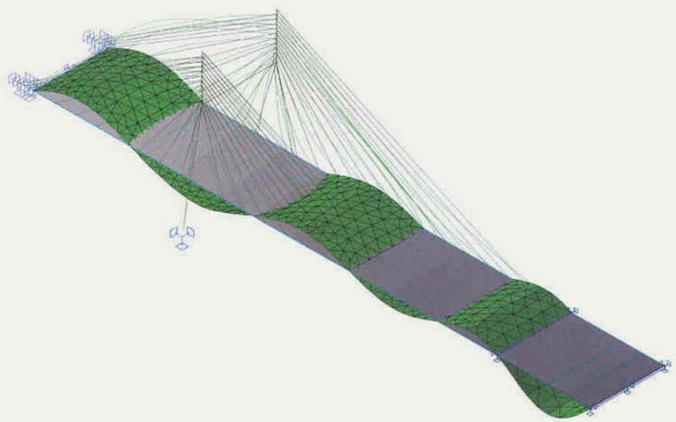
7.Vlastní tvar - vlastní frekvence $f_7 = 2{,}153$ Hz



9.Vlastní tvar - vlastní frekvence $f_9 = 2{,}522$ Hz



8.Vlastní tvar - vlastní frekvence $f_8 = 2{,}186$ Hz



10.Vlastní tvar - vlastní frekvence $f_{10} = 2{,}763$ Hz